A New Generalized Exponential Distribution: Application to Survival Data

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UNIOSUN JOURNAL OF BASIC AND APPLIED SCIENCES
OSUN STATE UNIVERSITY, OSOGBO
UJBAS

ISSN: 3121-9837
www.ujws.uniosun.edu.ng/ujbas
ujbas@uniosun.edu.ng

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History:

Volume 1, Number 1
Published: 10/05/2026

Keywords:
alpha power Harris exponential distribution, incomplete moments, characteristics function.

ABSTRACT

Modeling lifetime data with diverse and complex hazard shapes remains a significant challenge in survival analysis, as many classical distributions, including the exponential distribution, are limited by their constant hazard rate structure and lack of flexibility. In an effort to address this limitation, various authors have come up with numerous generalizations of the exponential distribution (E) in the literature. Likewise, this study developed a new generalization of the E distribution called alpha power Harris exponential (APHE) distribution. The APHE has four parameters, flexible distribution functions, and a non-monotone hazard rate function. Basic statistical properties of the proposed distribution including moments, incomplete moments, characteristic function, moment generating function, and Rényi entropy were derived and studied. Parameters of the model are estimated using the maximum likelihood estimation method. The usefulness of the APHE model in modeling lifetime data was demonstrated by fitting the model to real survival datasets and comparing its performance against some other competing distributions. The findings revealed that the APHE distribution provides a superior fit to the datasets considered in this study, based on standard goodness-of-fit criteria such as the Akaike information criterion (AIC), Bayesian information criterion (BIC), and Kolmogorov-Smirnov statistic. APHE distribution was found to be a flexible and promising alternative for modeling complex lifetime data in survival analysis, reliability engineering, and related fields.

1. INTRODUCTION

Over several decades, the development of new families of statistical distributions has undergone substantial progress in fields such as engineering, seismology, insurance, demography, and many others (Bashiru et al., 2023). Initially, research was based on the use of baseline distributions, such as the normal, exponential, Fréchet, negative binomial, and Poisson models (Ogunde et al., 2021). However, these standard probability models often provide a poor fit when used to model lifetime data, because of the growing complexity and varying characteristics of the data sets (Aljohani et al., 2025). As the challenges encountered by domains using lifetime datasets, particularly in areas that handled skewed and heavy-tailed data sets, grew more complex, researchers acknowledged the necessity to create more flexible probabilistic models (Makubate & Musekwa, 2024). Researchers have widely used the exponential distribution in survival data analysis due to its tractability and memoryless property. In contrast, the distribution's constant hazard rate limits its ability to model lifetime data with different hazard shapes. This is especially important for survival analysis, where non-monotonic

| failure rates are common (Christen & Rubio, 2024). Due to this limitation, a substantial body of literature has generalized, extended, and modified the exponential distribution to improve its flexibility and broaden its applicability to complex lifetime data across survival analysis, reliability engineering, insurance, and environmental sciences (Afify et al., 2018).

For Instance, Hussain et al. (2022) introduced the new generalized exponential extended exponentiated-exponential (NGE3-Exponential) distribution using the transformed-transformer method. The model's efficacy was confirmed through validation against both guinea pig and breast cancer survival datasets. Likewise, Ozkan and Simsek (2023) presented a generalized Marshall-Olkin exponentiated exponential distribution, they established its statistical characteristics and showcased its practical application using analyses of four different lifetime datasets.

Among other prominent recent developments, are the type I half-logistic exponential studied by Almarashi et al. (2018), the alpha power transformed extended exponential distribution by Alghamedi et al. (2020), the new extended generalized inverse exponential by Sule (2021), the power-modified Kies exponential distribution by Afify et al. (2022), the odd Lomax generalized exponential distribution by Sapkota & Kumar (2022), the transformed MG-extended exponential by Menberu & Goshu (2024), the inverse unit exponential by Alsadat et al. (2024), the exponentiated generalized Weibull exponential by Klakattawi (2025), and the new exponentiated. exponential (NEE) by Mushtaq et al. (2025), the exponentiated inverse exponential distribution by Mushtaq et al. (2025), the new odd-type exponential distribution by Sapkota et al. (2025), Ramos Louzada Exponential model by Aljohani et al. (2025).

In this study, a new four-parameter alpha power Harris exponential (APHE) distribution was introduced. As a new class of statistical models, its unique four-parameter structure allows it to adapt to diverse data shapes, and this makes it an ideal candidate for modelling complex survival data sets. The APHE distribution is capable of modelling constant, reversible, decreasing, increasing, reversed J-shaped, and inverted bathtub-shaped rates. It is a highly flexible and tractable distribution that is capable of modelling skewed and censored data sets that are frequently encountered in clinical trials and often elude existing extensions of the exponential distribution. Likewise, a review of the mathematical properties of the new APHE distribution was done. The maximum likelihood estimation (MLE) method was used to estimate the $APHE$ parameters, and an extensive Monte Carlo simulation study was carried out to assess and compare the performance of the estimators. Also, we demonstrated the practical relevance of $APHE$ in applied research by applying it to two real survival datasets, and it was found to have consistently outperformed four other competing lifetime distributions.

The rest of this article is structured as follows: Section 2 presents the derivation of the PDF, CDF, survival and hazard functions of the alpha power Harris Burr XII distribution and some other relevant mathematical expansions. Likewise, Section 2 presents the statistical properties of the distribution and describes the estimation method used. Section 3 reports the results of the analysis, which includes descriptive statistics, Monte Carlo simulation results, and the application of the APHBXII model to two survival data sets. In Section 4, the discussion of findings was presented, and Section 5 presents the conclusion of the study.

2. MATERIALS AND METHODS

2.1 Alpha Power Harris Exponential distribution

The cumulative distribution (CDF) of the E distribution is given by

$$J(x;\varphi) = 1 - e^{-\varphi x}, \qquad x > 0, \varphi > 0 \tag{1}$$

The associated probability density function (PDF) and survival function (SF) to (1) are given, respectively, by

$$j(x;\varphi) = \varphi e^{-\varphi x}, \qquad x > 0 \tag{2}$$

and

$$\bar{J}(x;\varphi) = e^{-\varphi x}, \qquad x > 0. \tag{3}$$

where $\varphi > 0$ is a rate parameter with mean $\frac{1}{\varphi}$. Using the PDF of the alpha power Harris family developed by Olalude et al. (2025) given as

$$f_{APH}(x) = \begin{cases} \dfrac{\log\alpha}{\alpha-1}\dfrac{c^{1/v} j(x;\Pi)}{(1-\bar{c}\bar{J}(x;\Pi)^{v})^{1+1/v}}\alpha^{\left\{1-\frac{c^{1/v}\bar{J}(x;\Pi)}{(1-\bar{c}\bar{J}(x;\Pi)^{v})^{1/v}}\right\}}, & if\ \alpha > 0, \alpha \neq 1 \\ G(x), & if\ \alpha = 0 \end{cases} \tag{4}$$

By inserting Equations (2) and (3) into Equation (4), this gives the PDF of the new alpha power Harris exponential $(APHE)$ distribution as

$$f_{APHE}(x) = \begin{cases} \dfrac{\varphi c^{1/v}\log\alpha e^{-\varphi x}}{(\alpha-1)(1-\bar{c}e^{-\varphi v x})^{1+1/v}}\alpha^{\left\{1-\frac{c^{1/v}e^{-\varphi x}}{(1-\bar{c}e^{-\varphi v x})^{1/v}}\right\}}, & if\ \alpha > 0, \alpha \neq 1 \\ G(x), & if\ \alpha = 1 \end{cases} \tag{5}$$

Its corresponding CDF to (5) is also derived as

$$F_{APHE}(x;\alpha,v,c,\varphi) = \begin{cases} \dfrac{\alpha^{\left\{1-\frac{c^{1/v}e^{-\varphi x}}{(1-\bar{c}e^{-\varphi v x})^{1/v}}\right\}}-1}{\alpha-1}, & if\ \alpha > 0, \alpha \neq 1 \\ 1-\dfrac{c^{1/v}e^{-\varphi x}}{(1-\bar{c}e^{-\varphi v x})^{1/v}}, & \alpha = 1 \end{cases}. \tag{6}$$

where α, v, c and φ are the parameters of the distribution.

2.2 Survival and hazard function of APHE distribution

The hazard function, $h(x)$ is considered to be the instant rate of failure at a given time t, while the survival function, $S(x)$ is the probability that subject under investigation will not experience any failure before time t. The survival function, $S(x)$, of the $APHE$ distribution is given by:

$$S(x;\alpha,\upsilon,c,\varphi) = \frac{\alpha - \alpha^{\left\{1-\frac{c^{1/\upsilon}e^{-\varphi x}}{(1-\bar{c}e^{-\varphi\upsilon x})^{1/\upsilon}}\right\}}}{\alpha - 1} \quad (7)$$

and the $h(x)$ of the $APHE$ distribution is of the form

$$h(x;\alpha,\upsilon,c,\varphi) = \frac{\varphi c^{1/\upsilon}\log(\alpha)\, e^{-\varphi x}\, \alpha^{\left\{1-\frac{c^{1/\upsilon}e^{-\varphi x}}{(1-\bar{c}e^{-\varphi\upsilon x})^{1/\upsilon}}\right\}}}{\left\{(1-\bar{c}e^{-\varphi\upsilon x})^{1+1/\upsilon}\right\}\left(\alpha - \alpha^{\left\{1-\frac{c^{1/\upsilon}e^{-\varphi x}}{(1-\bar{c}e^{-\varphi\upsilon x})^{1/\upsilon}}\right\}}\right)}. \quad (8)$$

Figure 1 present different shapes of the CDF, PDF, and Figure 2 display $s(x)$, and $h(x)$ curves of the newly developed $APHE$ distribution based on different values of its parameters.

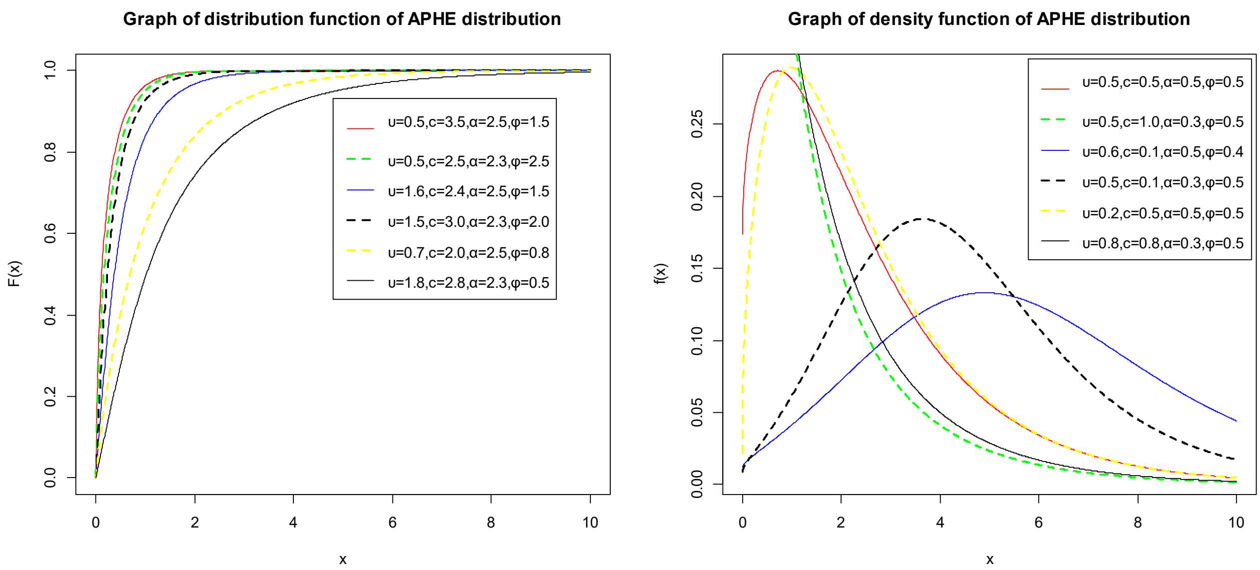

Figure 1. Plots of the CDF's and PDFs of the $APHE$ distribution

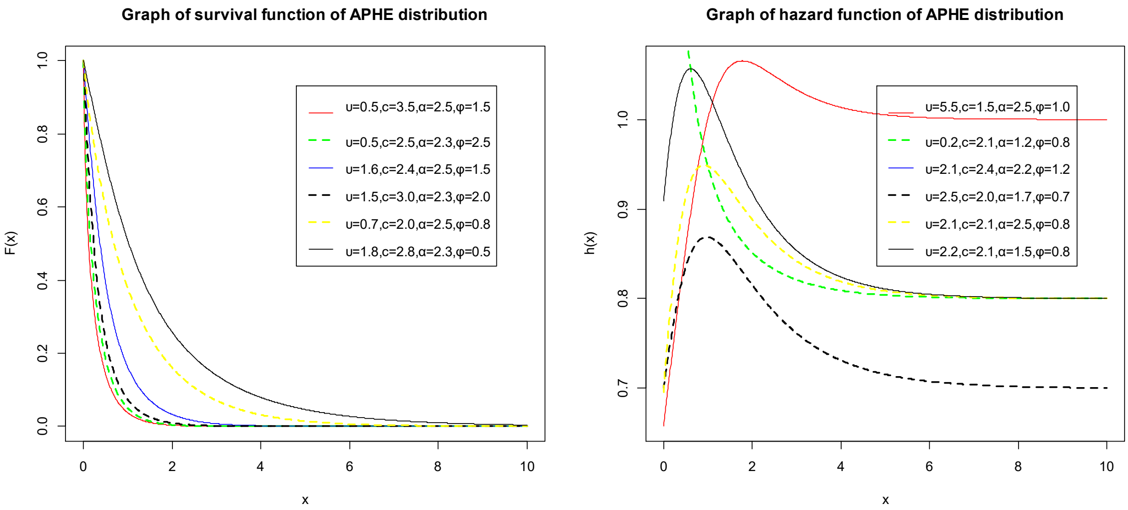

Figure 2. Plots of the survival and hazard functions of the $APHE$ distribution

2.3 Useful expansions

Given any real number d and $|h| < 1$, we have the following binomial series expansion

$$(1-h)^d = \sum_{p=0}^{\infty} (-1)^p \binom{d}{p} h^p, \qquad |h| < 1 \tag{9}$$

and if $h > 0$, then

$$h^d = e^{d \log h} = \sum_{p=0}^{\infty} \frac{(\log h)^p}{p!} d^p \tag{10}$$

Using the power series given in Equations (9) and (10), we can summarize (5) as

$$f_{APHE}(x) = \sum_{p=0}^{\infty} \sum_{r=0}^{\infty} a_{p,r}\, e^{-\varphi(p+rv+1)x} \tag{11}$$

Where $\bar{c} = 1 - c$,

$$a_{p,r} = \frac{\alpha\varphi}{(\alpha-1)} (-1)^p \frac{(\log\alpha)^{p+1}}{p!} c^{(p+1)/v} \binom{\frac{1+p}{v}+r}{r} (\bar{c})^r$$

Let $g(x;\lambda) = \lambda e^{-\lambda x}$ be the exponential density with rate λ, and define $\lambda_{p,r} = \varphi(p+rv+1)$, then Equation (11) can be written as an infinite linear combination of exponential densities:

$$f_{APHE}(x) = \sum_{p=0}^{\infty} \sum_{r=0}^{\infty} z_{p,r}\, g(x;\lambda_{p,r}), \tag{12}$$

where,

$$z_{p,r} = \frac{\alpha}{(\alpha-1)}(-1)^p \frac{(log\alpha)^{p+1}}{p!} c^{(p+1)/_v} \binom{\frac{1+p}{v}+r}{r} (\bar{c})^r \frac{1}{p+1+rv}$$

Therefore, integrating Equation (12), the CDF of the $APHE$ is

$$\mathbf{F}^{\mathbf{APHE}}(x) = \sum_{p=0}^{\infty}\sum_{r=0}^{\infty} z_{p,r}\, G(x;\lambda_{p,r}), \qquad (13)$$

where, $G(x;\lambda) = 1 - e^{-\lambda x}$ is the exponential CDF with rate λ.

2.4 Quantile function (QF)

The QF of $APHE$ function can be determined using

$$x_u = -\frac{1}{\varphi}\left\{ log\left[\frac{1-\frac{\log(1+u(\alpha-1))}{\log(\alpha)}}{\left\{c+\bar{c}\left(1-\frac{\log(1+u(\alpha-1))}{\log(\alpha)}\right)^{v}\right\}^{1/_v}}\right]\right\} \qquad (14)$$

It should be noted that u is uniformly random variable distributed between 0 and 1. Important quantile measures can be obtained from Equation (14), in particular, lower, middle (median), and the upper quartiles of X are respectively obtained by taken $u = 0.25, 0.5$ and 0.75 as follows:

$$x_{0.25} = -\frac{1}{\varphi}\left\{ log\left[\frac{1-\frac{\log(1+(\alpha-1)0.25}{\log(\alpha)}}{\left\{c+\bar{c}\left(1-\frac{\log(1+(\alpha-1)0.25}{\log(\alpha)}\right)^{v}\right\}^{1/_v}}\right]\right\}, \qquad (15)$$

$$x_{0.5} = -\frac{1}{\varphi}\left\{ log\left[\frac{1-\frac{\log(1+(\alpha-1)0.5}{\log(\alpha)}}{\left\{c+\bar{c}\left(1-\frac{\log(1+(\alpha-1)0.5}{\log(\alpha)}\right)^{v}\right\}^{1/_v}}\right]\right\}, \qquad (16)$$

and

$$x_{0.75} = -\frac{1}{\varphi}\left\{ log\left[\frac{1-\frac{\log(1+(\alpha-1)0.75}{\log(\alpha)}}{\left\{c+\bar{c}\left(1-\frac{\log(1+(\alpha-1)0.75}{\log(\alpha)}\right)^{v}\right\}^{1/_v}}\right]\right\} \qquad (17)$$

2.5 Moments

Let $X \sim APHE(\alpha, v, c, \varphi)$ distribution. Using the exponential-mixture representation in Equation (12), then the s^{th} ordinary moment is

$$\mu'_s = E(X^s) = \int_0^\infty x^s f_{APHE}(x)\,dx = \sum_{p=0}^\infty \sum_{r=0}^\infty z_{p,r} \int_0^\infty x^s \lambda_{p,r} e^{-\lambda_{p,r}x}\,dx \qquad (18)$$

Taking $l = \lambda_{p,r}x, x = \frac{l}{\lambda_{p,r}}, dx = \frac{dl}{\lambda_{p,r}}$, and substitute into Equation (18), where $\lambda_{p,r} = \varphi(p+1+rv)$. Hence, we have

$$\mu'_s = \Gamma(s+1) \sum_{p=0}^\infty \sum_{r=0}^\infty \frac{z_{p,r}}{[\varphi(p+1+rv)]^s} \qquad (19)$$

Where $z_{p,r}$ is define in Equation (12), and $\Gamma(s+1) = \int_0^\infty m^s e^{-m} dm$ is a complete gamma function. An expression for the first $(s=1)$, and the second moment $(s=2)$ are respectively, given by

$$\mu'_1 = E(X) = \sum_{p=0}^\infty \sum_{r=0}^\infty \frac{z_{p,r}}{\varphi(p+1+rv)} \qquad (20)$$

and

$$\mu'_2 = E(X^2) = 2\sum_{p=0}^\infty \sum_{r=0}^\infty \frac{z_{p,r}}{[\varphi(p+1+rv)]^2} \qquad (21)$$

The variance, $[\sigma^2 = \mu'_2 - (\mu'_1)^2]$ is given as

$$\sigma^2 = \left[2\sum_{p=0}^\infty \sum_{r=0}^\infty \frac{z_{p,r}}{[\varphi(p+1+rv)]^2} - \left(\sum_{p=0}^\infty \sum_{r=0}^\infty \frac{z_{p,r}}{\varphi(p+1+rv)}\right)^2\right] \qquad (22)$$

2.6 Incomplete moment of APHE distribution

Let $X \sim APHE(\alpha, v, c, \varphi)$, the s^{th} incomplete (raw) moment of up to $t > 0$ is obtained as follows:

$$\Gamma_s(t) = E(X^s) = \int_0^t x^s f_{APHE}(x)dx \qquad (23)$$

Using $f_{APHE}(x)dx$ as in Equation (12) we obtain

$$\Gamma_s(t) = \sum_{p=0}^\infty \sum_{r=0}^\infty z_{p,r} \int_0^t x^s \lambda_{p,r} e^{-\lambda_{p,r}x}\,dx \qquad (24)$$

Taking $m = \lambda_{p,r}x, x = \frac{m}{\lambda_{p,r}}, dx = \frac{dm}{\lambda_{p,r}}$, and substitute into Equation (24), finally we have

$$\Gamma_s(t) = \sum_{p=0}^\infty \sum_{r=0}^\infty z_{p,r} \frac{\Gamma(s+1) - \Gamma[(s+1), \varphi(p+1+rv)t]}{[\varphi(p+1+rv)]^s} \qquad (25)$$

An expression for the first ($s = 1$),

$$\Gamma_1(t) = \sum_{p=0}^{\infty}\sum_{r=0}^{\infty} z_{p,r}\frac{\Gamma(2) - \Gamma[2,\varphi(p+1+rv)t]}{\varphi(p+1+rv)} \qquad (26)$$

where $\Gamma(u,v) = \int_v^{\infty} l^{u-1}e^{-l}dl$ is define as the complementary incomplete gamma function. The expression given in Equation (26), is an important tool that can be used in obtaining an expression for the Bonferroni and Lorenz curves.

2.7 The moment generating function, (MGF) and characteristic function (CF)

Using the series expansion given by $e^{tX} = \sum_{s=0}^{\infty} t^s X^s /_{s!}$, an expression for the mgf can be derived using the s^{th} moment of the $APHE$ distribution as

$$M_X(t) = E(e^{tX}) = \sum_{s=0}^{\infty}\frac{t^s}{s!}E(X^s) \qquad (27)$$

After transformation, we obtained the equivalent power-series form

$$M_X(t) = \sum_{s=0}^{\infty} t^s \sum_{p=0}^{\infty}\sum_{r=0}^{\infty}\frac{z_{p,r}}{[\varphi(p+1+rv)]^s} \qquad |t| < \varphi$$

Where $z_{p,r}$ is defined in Equation (12). Similarly, we can derive an expression for the cf of the $APHE$ distribution using the s^{th} moment of the $APHE$ distribution as

$$\Omega_x(t) = E\left(e^{itX}\right) = \sum_{s=0}^{\infty}\frac{(it)^s}{s!}E(X^s), \qquad (28)$$

$$\Omega_x(t) = \sum_{s=0}^{\infty}(it)^s \sum_{p=0}^{\infty}\sum_{r=0}^{\infty}\frac{z_{p,r}}{[\varphi(p+1+rv)]^s} \qquad (29)$$

2.8 Rényi and β-entropies of APHE distribution

The Rényi entropy developed by Rényi (1959), is an important generalization of Shannon's entropy (Shannon, 1948). Entropy is a measure of degree of randomness of a system. Furthermore, the theory has been effectively applied in a number of fields such as, reliability, and physics (Song, 2001). In statistics, it is useful for testing hypotheses in lifetime distributions. Mathematically, the Rényi entropy is defined as:

$$\mathcal{R}_{\varphi}(X) = \frac{1}{1-\varphi}log\{Z(\varphi)\} \qquad \varphi > 0, \varphi \neq 1, \qquad (30)$$

where $Z(\varphi) = \int_0^{\infty} f_{APHE}(x)^{\varphi}dx, \varphi > 0$ and $\varphi \neq 1$. Using the PDF in Equation (5),

$$Z(\varphi) = \int_0^{\infty} \left(\frac{\varphi c^{1/v} log\alpha e^{-\varphi x}}{(\alpha - 1)(1 - \bar{c}e^{-\varphi vx})^{1+1/v}} \alpha^{\left\{1 - \frac{c^{1/v} e^{-\varphi x}}{(1-\bar{c}e^{-\varphi vx})^{1/v}}\right\}} \right)^{\varphi} dx \quad (31)$$

Also, by applying eqns. (9) and (10), and applying the integrand, we obtain

$$Z(\varphi) = \phi^{\varphi-1} \left(\frac{c^{1/v} \log(\alpha)}{(\alpha - 1)} \right)^{\varphi} \alpha^{\varphi} \sum_{p=0}^{\infty} \sum_{r=0}^{\infty} \frac{(-1)^p (\varphi log\alpha)^p}{p!} c^{p/v} (\bar{c})^r \frac{1}{\varphi + p + rv}$$

$$\binom{\varphi \left(1 + \frac{1}{v}\right) + \frac{p}{v} + r - 1}{r} \quad (32)$$

Taking $Z(\varphi)$ in Equation (32) and inserting in Equation (30), we obtain an expression for the Rényi entropy of the $APHE$ distribution as

$$\mathcal{R}_{\varphi}(X) = \frac{1}{1 - \varphi} log\{Z(\varphi)\} \quad (33)$$

Consequently, we obtained an expression for the β-entropy using the relation put forward by Tsallis (1988) as

$$T_{\varphi}(X) = \frac{1}{\varphi - 1} \left\{ 1 - \int_0^{\infty} f_{APHE}(x)^{\varphi} \right\}, \quad \varphi > 0, \varphi \neq 1, \quad (34)$$

Taking $H(\varphi)$ in Equation (32) and inserting in Equation (34), we have an expression for the β entropy of $APHE$ distribution as

$$T_{\varphi}(X) = \frac{1}{\varphi - 1} \{1 - \{H(\varphi)\}\} \quad (35)$$

2.9 Maximum likelihood estimation (MLE) of APHE model

MLE is a method that estimates the values of a statistical model's unknown parameters by maximizing the likelihood function of the distribution. Mathematically, given a sample of data $X_N = x_{N1}, x_{N2}, x_{N3}, \dots, x_{Nn}$ and a probability distribution $f(X_N/\Omega_N)$ where Ω_N are unknown parameters, the likelihood function is represented as

$$L(\Omega_N, x_N) = \prod_{i=1}^{n} f_{APHE}(x_{Ni}; \Omega_N) \quad (36)$$

For $APHE$, by putting Equation (5) in Equation (36), we have

$$L = \prod_{i=1}^{n} \left\{ \frac{\varphi c^{1/v} log\alpha e^{-\varphi x}}{(\alpha - 1)(1 - \bar{c}e^{-\varphi vx})^{1+1/v}} \alpha^{\left\{1 - \frac{c^{1/v} e^{-\varphi x}}{(1-\bar{c}e^{-\varphi vx})^{1/v}}\right\}} \right\} \quad (37)$$

The log-likelihood ($l(\Omega_N, x_N) = logL$) is

$$l(\Omega_N, x_N) = n\log\left(\frac{\varphi c^{1/v}\log(\alpha)}{(\alpha-1)}\right) - \varphi\sum_{i=1}^{n} x_i - (1 + {}^{1}/_{v})\sum_{i=1}^{n} \log(1 - \bar{c}e^{-\varphi v x_i})$$

$$+\log(\alpha)\sum_{i=1}^{n}\left\{1 - \frac{c^{1/v} e^{-\varphi x_i}}{(1-\bar{c}e^{-\varphi v x_i})^{1/v}}\right\} \quad (38)$$

The ML estimates of the parameters, were derived by obtaining solutions to the non-linear equations in Equation (38). Since the above expression in Equation (38) is difficult to calculate analytically, we adopted a numerical solution method to determine the values of the unknown parameters (α, v, c, φ) using the $AdequacyModel$ library in R programming.

3. RESULTS

3.1 Descriptive statistics

Table 1 gives the first six moments, standard deviation (σ), coefficient of variation (CV), skewness and kurtosis of the $APHE$ distribution.

Table 1. First six moments, CV, skewness, and kurtosis of $APHE$ distribution

$\varphi = 2.5$ and c= 0.5

Moments	$v = 3.0$	$v = 3.5$	$v = 0.5$	$v = 5.5$	$v = 8.0$
	$\alpha = 1.5$	$\alpha = 0.5$	$\alpha = 2.5$	$\alpha = 3.5$	$\alpha = 5.5$
μ'_1	0.3726	0.2794	0.3096	0.4862	0.5467
μ'_2	0.3000	0.1982	0.1994	0.4366	0.5147
μ'_3	0.3644	0.2271	0.2037	0.5569	0.6715
μ'_4	0.5877	0.3560	0.2917	0.9194	1.1218
μ'_5	1.1806	0.7051	0.5415	1.8687	2.2941
μ'_6	2.8403	1.6845	1.2372	4.5218	5.5689
σ	0.4015	0.3467	0.3217	0.4475	0.4646
CV	1.0775	1.2409	1.0393	0.9205	0.8499
$S\kappa_w$	2.0470	2.5085	2.3400	1.6718	1.5371
$S\kappa_u$	9.1107	12.2360	11.8013	7.1834	6.6184

3.2 Simulation study

This study conducted a Monte Carlo simulation to examine the finite-sample behavior of the MLEs for the parameters of the APHE model. The simulation was carried out for different sample sizes, n = 50, 100, 300, 500, 700 and 1000 for two distinct sets of parameter values: Set 1 $(v = 0.5, \alpha = 0.3, c = 0.95, \varphi = 2.5)$ and Set 2 $(v = 0.5,\ \alpha = 0.25,\ c = 0.9,\ \varphi = 2.0)$. For each combination of n and the true parameter values, the simulation was replicated $R = 1000$ times. The results of the simulation experiment are given in Table 2, containing the Absolute value (AB), standard error (σ) and the mean squared error (MSE).

Table 2. Estimate, SE, and MSEs of $APHE$ distribution

	n	AB	σ	MSE		n	AB	σ	MSE
	50	6.916	28.093	835.863		50	2.979	11.777	147.372
	100	4.143	18.421	356.062		100	2.232	10.042	105.689
$v = 0.5$	300	2.916	13.470	216.642	$v = 0.5$	300	1.980	9.301	92.131
	500	2.173	8.519	77.221		500	1.728	8.560	78.572
	700	1.999	7.838	71.044		700	1.459	7.819	65.013
	1000	1.839	7.211	65.360		1000	1.190	7.077	51.454
	50	4.374	36.588	1355.871		50	2.199	9.690	98.597
	100	1.624	7.151	53.707		100	1.496	6.862	49.258
$\alpha = 0.3$	300	0.567	2.925	8.866	$\alpha = 0.25$	300	0.602	2.424	6.230
	500	0.409	1.213	1.637		500	0.447	1.353	2.028
	700	0.302	0.887	0.878		700	0.350	1.028	1.179
	1000	0.179	0.474	0.256		1000	0.288	0.891	0.875
	50	8.854	76.612	5939.194		50	5.153	12.024	170.921
	100	3.489	8.523	84.709		100	3.677	8.535	86.268
$c = 0.95$	300	2.480	6.072	42.976	$c = 0.9$	300	3.014	6.513	51.462
	500	1.811	4.772	26.025		500	1.987	5.009	29.006
	700	1.436	3.770	16.255		700	1.861	4.443	23.517
	1000	1.143	3.358	12.568		1000	1.736	3.877	18.029
	50	7.523	18.310	391.356		50	6.742	15.965	299.971
	100	5.951	14.670	250.342		100	5.137	12.486	182.096
$\varphi = 2.5$	300	4.132	10.211	121.218	$\varphi = 2.0$	300	3.625	8.653	87.940
	500	2.933	7.973	72.099		500	2.308	6.615	49.042
	700	2.458	6.935	54.086		700	2.114	6.318	46.268
	1000	1.774	5.623	34.729		1000	1.920	5.093	29.604

3.3 Application to survival data

In this study, the $APHE$ distribution was applied to two lifetime data to show the flexibility and applicability of the proposed model. Also, the distribution density was fitted and compared to some of its sub-models, that includes Harris exponential (HE), Marshall-Olkin exponential (MOE), alpha power exponential (APE), and exponential (E) distribution. The PDFs of the competing models were given as:

$$f_{HE}(x;\upsilon,c,\varphi) = \frac{\varphi c^{1/\upsilon} e^{-\varphi x}}{(1-\bar{c}e^{-\varphi\upsilon x})^{1+1/\upsilon}}, \qquad x>0;\varphi,c,\upsilon>0 \qquad (39)$$

$$f_{MOE}(x;c,\varphi) = \frac{c\varphi e^{-\varphi x}}{(1-\bar{c}e^{-\varphi x})^{2}}, \qquad x>0;\varphi,c>0 \qquad (40)$$

and

$$f_{APE}(x;\alpha,\varphi) = \frac{\log\alpha\ \varphi e^{-\varphi x}}{(\alpha-1)}\alpha^{\{1-e^{-\varphi x}\}}, \qquad x>0;\alpha,\varphi>0,\alpha \neq 1 \qquad (41)$$

$$f_{E}(x;\varphi) = \varphi e^{-\varphi x}, \qquad x>0;\ \varphi>0 \qquad (42)$$

Several goodness-of-fit measures were considered to assess the adequacy of the fitted model, including the Deviance ($neg2l$), Akaike Information Criterion ($AICr$), Consistent Akaike Information Criterion ($CAICr$), Kolmogorov Smirnoff (KS), and the corresponding probability value (P). The smaller these statistics are, the better the model fits the data except for the P value among the competing models. The study also provided the standard errors and confidence interval for the estimated parameters of the model in parentheses and curly brackets respectively.

The first data set represents the remission times (in months) of a random sample of 128 bladder cancer patients. For previous study see Lee and Wang (2003).

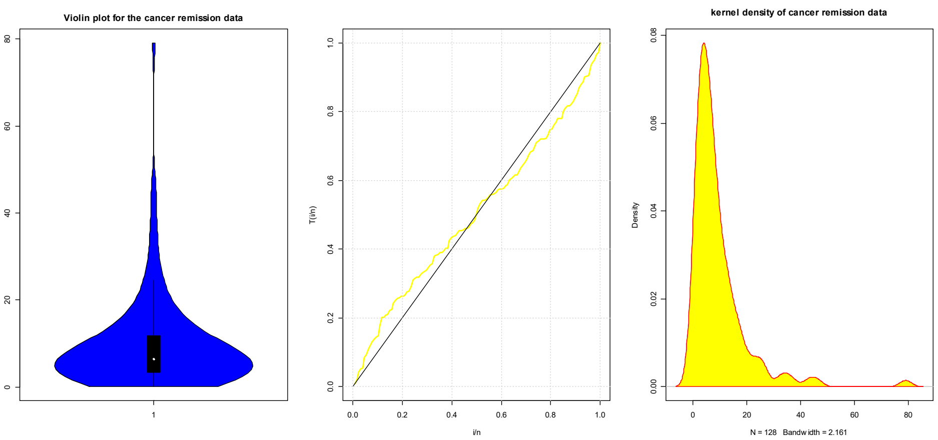

Figure 3. Violin, Total Time on Test, and Kernel density plot for cancer data

Table 3. MLE's, standard error, 95% confidence interval, and measures of goodness-of-fit

	α	c	φ	v	$neg2l$	$AICr$	$CAICr$	KS	pv
APHE	2.4445 (0.7407) {0.9928, 3.8962}	0.0746 (0.1154) {-0.1517, 0.3008}	0.0711 (0.0274) {0.0175, 0.1248}	3.5196 (2.2141) {-0.8201, 7.8592}	821.30	829.30	829.62	0.060	0.7548
HE	-	2.8715 (1.6433) {-0.3495, 6.0925}	0.1173 (0.0117) {0.0942, 0.1403}	9.9963 (6.5093) {-2.7619, 22.7545}	823.91	829.91	830.11	0.067	0.6211
APE	1.7942 (1.3853) {-0.9209, 4.5093}	-	0.1237 (0.0243) {0.0760, 0.1714}	-	829.03	833.03	833.12	0.074	0.4833
ME	-	1.0535 (0.3210) {0.4244, 1.6826}	0.1097 (0.0199) {0.0708, 0.1486}	-	828.65	832.65	832.75	0.081	0.3669
E	-	-	0.1069 (0.0094) {0.0884, 0.1254}	-	828.68	830.68	830.72	0.085	0.3157

The survival periods (measured in days) of 72 guinea pigs exposed to virulent tubercle bacilli are represented in the second data set, as obtained from Bjerkedal (1960).

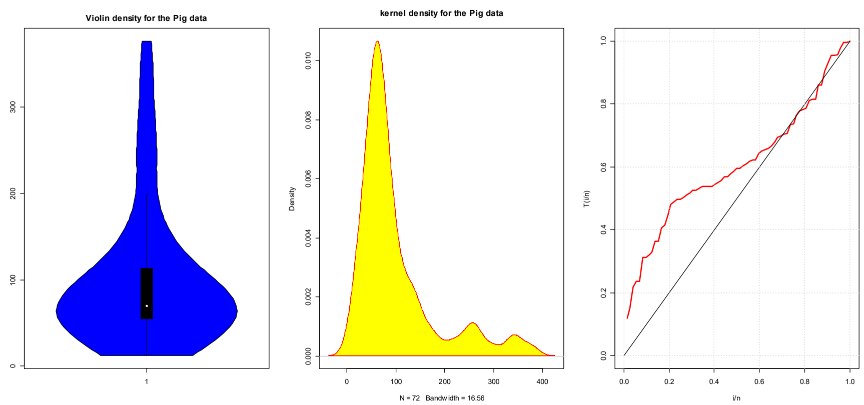

Figure 4. Violin, Total Time on Test, and Kernel density plot for Guinea pig data

Table 4. MLE's, standard error, 95% confidence interval, and measures of goodness-of-fit

	α	c	φ	v	$neg2l$	$AICr$	$CAICr$	KS	pv
APHE	2.2761 (0.8651)	9.6011 (3.2104)	0.0185 (0.0013)	3.8231 (1.9820)	791.42	799.42	800.01	0.137	0.1348
	{0.5805, 3.9717}	{3.3087, 15.8935}	{0.0159, 0.0211}	{-0.0617, 7.7080}					
HE	-	2.6006 (1.5266)	0.0154 (0.0025)	1.2935 (1.4085)	797.42	803.42	803.77	0.142	0.1102
		{-0.3915, 5.5927}	{0.0105, 0.0203}	{-1.4671, 4.0541}					
APE	2.4891 (1.2801)	-	0.0127 (0.0020)	-	800.96	804.96	805.13	0.178	0.0206
	{-0.0199, 4.9981}		{0.0088, 0.0167}						
ME	-	2.2713 (0.7759)	0.0152 (0.0027)	-	799.44	803.44	803.61	0.152	0.0726
		{0.7505, 3.7920}	{0.0099, 0.0205}						
E	-	-	0.0100 (0.0012)	-	806.88	808.88	808.94	0.212	0.0031
			{0.0077, 0.0123}						

4. DISCUSSION

The graphical representations of the (PDF) and (CDF) in Figure 1. demonstrate the flexibility of the APHE distribution in capturing various skewness and kurtosis levels of lifetime data. Furthermore, the analysis of the survival function and hazard rate function (HRF) in Figure 2. demonstrates that the model can exhibit increasing, decreasing, reversed-J, and inverted bathtub failure rate shapes. This versatility indicates that the APHE is well-suited for modeling diverse lifetime datasets, regardless of the underlying hazard rate behavior.

Table 1. presents the moments, CV, skewness, and kurtosis of the APHE distribution, computed for fixed φ = 2.5 and c = 0.5 across five combinations of v and α. The results showed that the APHE distribution is consistently right-skewed and leptokurtic, with moderate relative dispersion. The findings further revealed that both the central tendency and tail heaviness increase as the shape parameters v and α take larger values. This confirmed that the APHE distribution is flexible enough to capture a wide range

of data behaviors.

The simulation results in Table 2 confirmed that the maximum likelihood method effectively estimated the parameters of the APHE distribution. As the sample size increases from n = 50 to n = 1000, the absolute bias, standard error, and mean squared error for all four parameters showed a steady decline. This downward trend validates the consistency of the estimators and indicates that the estimates get closer to the true values as more data becomes available. Among the parameters, α showed the most precise results, with its bias dropping by over 90% in most cases. Although the shape parameter υ converges more slowly due to its influence on the distribution's tail behavior, this is a common challenge noted in similar statistical literature (Warahena-Liyanage et al., 2023). These findings match recent studies on similar alpha power families such as Eghwerido et al. (2023), which also found that larger samples lead to highly reliable estimates.

In the empirical applications, the descriptive analysis showed that both the bladder cancer and guinea pig datasets were over-dispersed, leptokurtic, and positively skewed. As demonstrated in Figure 3, the cancer data exhibited an increasing failure rate, while Figure 4. indicated a non-monotone failure rate for the guinea pig data. Based on the comparative results presented in Table 3, it was concluded that the APHE model provided the superior fit for the cancer dataset, as evidenced by the smallest values for the Deviance (neg2l), Akaike Information Criterion (AICr), corrected AICr (CAICr), and Kolmogorov-Smirnov (KS) statistic, alongside the largest p-value (P). Similarly, the results in Table 4. also established that the APHE was the best-fitting model for the guinea pig data, outperforming all competing distributions across the evaluated goodness-of-fit metrics.

5. CONCLUSION

This study presented the development of a novel four-parameter model known as the alpha power Harris exponential distribution, and some of its relevant statistical properties were discussed in details. The addition of the alpha power parameter gave extra flexibility to the baseline distribution and enabled the model to cover a wider set of hazard shapes and tail behaviors that the Harris exponential alone cannot capture. The Monte Carlo simulations used different sample sizes to examine the theoretical performance of the MLE estimators. To evaluate the precision and applicability of the estimations, the distribution was applied to two survival datasets. When applied to both datasets, the alpha power Harris exponential (APHE) distribution outperformed its sub-models. This confirmed that the extension of the alpha power component to the already existing Harris exponential distribution was not redundant, as it meaningfully improved the fit and overall performance of the model. It is expected that the APHE model will be used in wider applications in fields such as the physical and biological sciences.

ACKNOWLEDGMENT

The author gratefully acknowledged all the sources of references and datasets used in this research.

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